**Budker Institute of Nuclear Physics**

**Budker INP 99-63**
**July 27, 1999**



# Quasiclassical Green function in an external field and small-angle scattering

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### Abstract

The quasiclassical Green functions of the Dirac and Klein-Gordon equations in the external electric field are obtained with the first correction taken into account. The relevant potential is assumed to be localized, while its spherical symmetry is not required. Using these Green functions, the corresponding wave functions are found in the approximation similar to the Furry-Sommerfeld-Maue approximation. It is shown that the quasiclassical Green function does not coincide with the Green function obtained in the eikonal approximation and has a wider region of applicability. It is illustrated by the calculation of the small-angle scattering amplitude for a charged particle and the forward photon scattering amplitude. For charged particles, the first correction to the scattering amplitude in the non-spherically symmetric potential is found. This correction is proportional to the scattering angle. The real part of the amplitude of forward photon scattering in a screened Coulomb potential is obtained.

# 1 Introduction

As known, the use of the wave functions and Green functions of the Dirac equation in an external field is a convenient tool for the calculation of QED amplitudes in the field. At high energy and small scattering angles, the characteristic angular momenta of involved particles are large, and the quasiclassical approximation is applicable. In this case the use of quasiclassical Green functions greatly simplifies the calculations.

The quasiclassical Green function $G(\mathbf{r}_2, \mathbf{r}_1|\varepsilon)$ of the Dirac equation in a Coulomb potential was first derived in [1, 2] starting from the exact Green function of the Dirac equation [3]. The more convenient representation of the quasiclassical Coulomb Green function for the case of almost collinear vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ was obtained in [4, 5]. In the same geometry the quasiclassical Green function in arbitrary spherically symmetric decreasing potential was found in [6, 7].

In section 2 we derive the quasiclassical Green functions of the Dirac and Klein-Gordon equations for the case of a localized potential not assumed to be spherically-symmetric. We use the term 'localized potential' for the decreasing potential having a maximum at some point. The Green functions are obtained with the first correction taken into account. With their help the quasiclassical wave functions of the Dirac and Klein-Gordon equations and corrections to them are found (section 3). These wave functions generalize the results obtained in the Furry-Sommerfeld-Maue approximation [8, 9, 10].

In the calculation of amplitudes of the high-energy processes the eikonal approximation is often used. The corresponding wave functions and Green functions differ, generally speaking, from the quasiclassical ones and have a narrower region of applicability. Therefore, the use of the eikonal approximation without a special consideration of its applicability may lead to incorrect results. As an example, we show this in section 4 for the problem of small-angle scattering of a charged particle in an external field. In this section we present also the consequent derivation of the expression for the scattering amplitude using the quasiclassical wave function. In particular case of a Coulomb potential the use of the eikonal wave function instead of the quasiclassical one leads to the incorrect result.

In section 5 the quasiclassical Green function obtained is used in the calculation of the amplitude of forward elastic scattering of a photon in the atomic electric field (forward Delbruck scattering). As shown in [2], at this calculation it is necessary to take into account the correction to the quasiclassical Green function. This correction should be taken into account in the region, where the eikonal approximation is valid. The contribution of higher orders of the perturbation theory with respect to the external field (Coulomb corrections) is determined by the region, where the eikonal Green function is inapplicable. The real part of the forward Delbruck scattering amplitude for a screened Coulomb potential is obtained. It becomes comparable with the amplitude of Compton scattering already at relatively small energies of a photon. It may be important for the description of photon propagation in matter.

## 2 Green function

As shown in papers [6, 4], in the calculation of amplitudes of various QED processes it is convenient to use the Green function of squared Dirac equation $D(\mathbf{r}_2, \mathbf{r}_1|\varepsilon)$, which is connected to conventional Green function $G(\mathbf{r}_2, \mathbf{r}_1|\varepsilon)$ by the relation

$$G(\mathbf{r}_2, \mathbf{r}_1|\,\varepsilon) = \Big[\gamma^0(\varepsilon \,-\, V(\mathbf{r}_2)) \,-\, \boldsymbol{\gamma}\,\mathbf{p}_2 \,+\, m\Big]\, D(\mathbf{r}_2, \mathbf{r}_1|\,\varepsilon)\,, \tag{1}$$

where $\gamma^\mu$ are the Dirac matrices, $\mathbf{p} = -i\boldsymbol{\nabla}$ is the momentum operator, and $V(\mathbf{r})$ is the potential. In the quasiclassical approximation it is possible to present function $D(\mathbf{r}_2, \mathbf{r}_1|\varepsilon)$ as

$$D(\mathbf{r}_2, \mathbf{r}_1|\,\varepsilon) = \left[1 \,-\, \frac{i}{2\varepsilon}(\boldsymbol{\alpha}\;, \boldsymbol{\nabla}_1 + \boldsymbol{\nabla}_2\,)\right] D^{(0)}(\mathbf{r}_2, \mathbf{r}_1|\,\varepsilon)\,, \tag{2}$$

where

$$D^{(0)}(\mathbf{r}_2, \mathbf{r}_1|\,\varepsilon) = \langle \mathbf{r}_2| \frac{1}{\kappa^2 \,-\, H \,+ i0}|\mathbf{r}_1\rangle \quad , \quad H = \mathbf{p}^2 + 2\varepsilon V(\mathbf{r}) - V^2(\mathbf{r}) \tag{3}$$

and $\kappa^2 = \varepsilon^2 - m^2$. Thus, the problem is reduced to the calculation of the quasiclassical Green function $D^{(0)}$ of the Klein-Gordon equation with potential $V(\mathbf{r})$ (Schrodinger equation with Hamiltonian $H$).

Let us pass in function $D^{(0)}(\mathbf{r}_2, \mathbf{r}_1|\,\varepsilon)$ from variables $\mathbf{r}_1$ and $\mathbf{r}_2$ to variables $\mathbf{r}_1$ and $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. In terms of these variables the function $D_0$ satisfies to the equation

$$[\kappa^2 - 2\kappa\phi(\mathbf{r}_1 + \mathbf{r}) - \mathbf{p}^2] D^{(0)}(\mathbf{r} + \mathbf{r}_1, \mathbf{r}_1|\,\varepsilon) = \delta(\mathbf{r})\;, \tag{4}$$

where $\phi = \lambda V - V^2/2\kappa$, $\lambda = \varepsilon/\kappa$, $\mathbf{p} = -i\boldsymbol{\nabla}_r$. In the ultrarelativistic case $\lambda = +1$ for $\varepsilon > 0$ and $\lambda = -1$ for $\varepsilon < 0$.

We seek for a solution to this equation in the form

$$D^{(0)}(\mathbf{r} + \mathbf{r}_1, \mathbf{r}_1|\,\varepsilon) = -\frac{\exp(i\kappa r)}{4\pi r} F(\mathbf{r}, \mathbf{r}_1)\;. \tag{5}$$

Note, that the factor in front of $F$ in (5) is a Green function of the equation (4) for $\phi = 0$. The function $F$ satisfies the equation

$$\left[i\frac{\partial}{\partial r} - \phi(\mathbf{r}_1 + \mathbf{r})\right] F = -\frac{1}{2\kappa} r\Delta(F/r) \tag{6}$$

with the boundary condition $F(\mathbf{r} = 0, \mathbf{r}_1) = 1$. For further needs we define an effective impact parameter $\rho_*$ of a rectilinear trajectory $\Gamma$, connecting point $\mathbf{r}_1$ and $\mathbf{r}_2$, as

$$\rho_* = \min_{\mathbf{x}\in\Gamma} \frac{|\phi(\mathbf{x})|}{|\boldsymbol{\nabla}_\perp \phi(\mathbf{x})|}\;. \tag{7}$$

Let this minimum is achieved at some point $\mathbf{x} = \mathbf{r}_*$. The necessary condition $\kappa\rho_* \gg 1$ for the applicability of the quasiclassical approximation is assumed to be fulfilled. Introducing the notation $a_{1,2} = |\mathbf{r}_{1,2} - \mathbf{r}_*|$, we have two overlapping regions:

$$\begin{aligned} &1)\ \min(a_1, a_2) \ll \kappa\rho_*^2 \\ &2)\ \min(a_1, a_2) \gg \rho_*\;. \end{aligned} \tag{8}$$

In the first region, the r.h.s. of the equation (6) is small. The solution to this equation with zero r.h.s. has the form

$$F_0 = \exp\left[-ir\int_0^1 \phi(\mathbf{r}_1 + x\mathbf{r})dx\right] \,, \tag{9}$$

which corresponds to the eikonal approximation. For the calculation of the first correction to $F_0$ we search for a solution of the equation (6) in the form $F = F_0(1+g)$ and neglect $g$ in r.h.s.. As a result, we obtain the following equation for $g$:

$$2i\kappa\frac{\partial}{\partial r}g = \phi^2(\mathbf{r}_1+\mathbf{r}) + ir\int\limits_0^1 dx\, x^2\Delta_1\phi(\mathbf{r}_1+x\mathbf{r}) + r^2\left[\int\limits_0^1 dx\, x\boldsymbol{\nabla}_{1\perp}\phi(\mathbf{r}_1+x\mathbf{r})\right]^2 \,, \tag{10}$$

where the index 1 at the derivative designates differentiation over $\mathbf{r}_1$, $\boldsymbol{\nabla}_{1\perp}$ is a component of a gradient, perpendicular to $\mathbf{r}$, $\Delta_1 = \boldsymbol{\nabla}_1^2$. Integrating over r we obtain for $g$

$$\begin{aligned} g &= \frac{1}{2\kappa}\left[r^2\int\limits_0^1 dx\, x(1-x)\Delta_1\phi(\mathbf{r}_1+x\mathbf{r}) - ir\int\limits_0^1 dx\,\phi^2(\mathbf{r}_1+x\mathbf{r}) - \right.\\ &- \left. 2ir^3\int\limits_0^1 dx(1-x)\boldsymbol{\nabla}_{1\perp}\phi(\mathbf{r}_1+x\mathbf{r})\int\limits_0^x dy\, y\boldsymbol{\nabla}_{1\perp}\phi(\mathbf{r}_1+y\mathbf{r})\right] \end{aligned} \tag{11}$$

Finally, up to the first correction, the Green function $D^{(0)}$ in the first region reads

$$\begin{aligned} D^{(0)}(\mathbf{r}_2,\mathbf{r}_1|\varepsilon) &= -\frac{1}{4\pi r}\exp\left[i\kappa r - i\lambda r\int_0^1 V(\mathbf{r}_1+x\mathbf{r})dx\right]\times \\ &\times \left\{1+\frac{1}{2\kappa}\left[\lambda r^2\int\limits_0^1 dx\, x(1-x)\Delta_1 V(\mathbf{r}_1+x\mathbf{r}) - \right.\right.\\ &- \left.\left. 2ir^3\int\limits_0^1 dx(1-x)\boldsymbol{\nabla}_{1\perp}V(\mathbf{r}_1+x\mathbf{r})\int\limits_0^x dy\, y\boldsymbol{\nabla}_{1\perp}V(\mathbf{r}_1+y\mathbf{r})\right]\right\} . \end{aligned} \tag{12}$$

In this expression we put $\lambda^2 = 1$ assuming $\varepsilon \gg m$. Substituting (43) in (2), we find the expression for function $D$ in the first region

$$\begin{aligned} &D(\mathbf{r}_2,\mathbf{r}_1|\varepsilon) = -\frac{1}{4\pi r}\exp\left[i\kappa r - i\lambda r\int_0^1 V(\mathbf{r}_1+x\mathbf{r})dx\right]\times \\ &\times\left\{1-\frac{r}{2\kappa}\int\limits_0^1 dx\,\boldsymbol{\alpha}\,\boldsymbol{\nabla}_1 V(\mathbf{r}_1+x\mathbf{r}) + \frac{\lambda r^2}{2\kappa}\int\limits_0^1 dx\, x(1-x)\Delta_1 V(\mathbf{r}_1+x\mathbf{r}) - \right.\\ &- \left.\frac{ir^3}{\kappa}\int\limits_0^1 dx(1-x)\boldsymbol{\nabla}_{1\perp}V(\mathbf{r}_1+x\mathbf{r})\int\limits_0^x dy\, y\boldsymbol{\nabla}_{1\perp}V(\mathbf{r}_1+y\mathbf{r})\right\} . \end{aligned} \tag{13}$$

In the second region the r.h.s. of the equation (6) is not small. Using spherical coordinates, we can rewrite (6) as

$$\left[i\frac{\partial}{\partial r} - \phi(\mathbf{r}_1+\mathbf{r}) - \frac{\mathbf{L}^2}{2\kappa r^2}\right]F = -\frac{1}{2\kappa}\frac{\partial^2}{\partial r^2}F \,, \tag{14}$$

where $\mathbf{L}^2$ is the angular momentum operator squared. In this equation $\mathbf{r}$ is a free variable. We are interested in a value of function $F(\mathbf{r},\mathbf{r}_1)$ at $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$. It is convenient to direct the polar axis along $\mathbf{r}_2-\mathbf{r}_1$. The term in the l.h.s. of (14) containing $\mathbf{L}^2$ should be taken into account for $r\geq a_1$. It can be checked by applying the operator $\mathbf{L}^2$ to the eikonal function (9). When calculating the function $F(\mathbf{r}_2-\mathbf{r}_1,\mathbf{r}_1)$ it is enough to consider a narrow region of polar angles of vector $\mathbf{r}$ where $\theta\sim\rho_*/a_1\ll 1$. The r.h.s. of (14) is small. We seek for a solution to this equation in the form

$$F=\mathrm{e}^{iA}\mathcal{F}\ ,\quad A=\left(\frac{1}{r}-\frac{1}{a_1}\right)\frac{\mathbf{L}^2}{2\kappa} \tag{15}$$

Substituting (15) in (14), we obtain $\mathcal{F}$

$$\left[i\frac{\partial}{\partial r}-\tilde{\phi}\right]\mathcal{F}=-\frac{1}{2\kappa}\left[\frac{\partial^2}{\partial r^2}-\frac{i\mathbf{L}^2}{\kappa r^2}\left(\frac{\partial}{\partial r}-\frac{1}{r}-\frac{i\mathbf{L}^2}{4\kappa r^2}\right)\right]\mathcal{F}\ , \tag{16}$$

where

$$\tilde{\phi}=\mathrm{e}^{-iA}\phi(\mathbf{r}_1+\mathbf{r})\mathrm{e}^{iA}\ . \tag{17}$$

We are going to solve this equation up to the first correction in the parameter $1/\kappa\rho_*$. For this purpose we should keep only two terms in the expansion of $\tilde{\phi}$ in terms of commutators of the operator $A$ and $\phi$: $\tilde{\phi}\approx\phi-i[A,\phi]$. In the first approximation it is possible to neglect r.h.s. of (16) and to replace $\tilde{\phi}$ by $\phi$. Then the function $\mathcal{F}$ coincides with the eikonal function $F_0$ (see (9)). To find the first correction, we present $\mathcal{F}$ as $\mathcal{F}=F_0(1+g_1)$. We obtain the following equation for $g_1$

$$2i\kappa\frac{\partial}{\partial r}g_1=\phi^2(\mathbf{r}_1+\mathbf{r})-\left(\frac{1}{r}-\frac{1}{a_1}\right)\left[i\mathbf{L}^2\phi(\mathbf{r}_1+\mathbf{r})+2r(\mathbf{L}\phi(\mathbf{r}_1+\mathbf{r}))\int\limits_0^1 dx\mathbf{L}\phi(\mathbf{r}_1+x\mathbf{r})\right]\ . \tag{18}$$

Integrating it over $r$, we find

$$\begin{aligned} g_1 \;=\; & -\frac{ir}{2\kappa}\Bigg[\int\limits_0^1 dx\phi^2(\mathbf{r}_1+x\mathbf{r})+ir\int\limits_0^1 dx\,x\left(1-\frac{r}{a_1}x\right)\Delta_{1\perp}\phi(\mathbf{r}_1+x\mathbf{r})+ \\ & +2r^2\int\limits_0^1 dx\left(1-\frac{r}{a_1}x\right)(\boldsymbol{\nabla}_{1\perp}\phi(\mathbf{r}_1+x\mathbf{r}))\int\limits_0^x dy\,y\boldsymbol{\nabla}_{1\perp}\phi(\mathbf{r}_1+y\mathbf{r})\Bigg]\ . \end{aligned} \tag{19}$$

Here $\Delta_{1\perp}=\boldsymbol{\nabla}_{1\perp}^2$. With the help of the expansion in spherical functions it can be shown, that at $\beta\ll 1$ for arbitrary function $g(\mathbf{r})$ the following relation is true with the appropriate accuracy:

$$\exp[-i\beta^2\mathbf{L}^2]\,g(\mathbf{r})\approx\int\frac{d\mathbf{q}}{i\pi}\mathrm{e}^{iq^2}\,g(\mathbf{r}+2\beta r\mathbf{q})\ , \tag{20}$$

where $\mathbf{q}$ is a two-dimensional vector, perpendicular to $\mathbf{r}$. Using (15), (19) and (20), we obtain in the second region the following expression for the Green function $D^{(0)}$ with the first correction taken into account:

$$D^{(0)}(\mathbf{r}_2,\mathbf{r}_1|\,\varepsilon)\;=\;\frac{i\mathrm{e}^{i\kappa r}}{4\pi^2 r}\int d\mathbf{q}\exp\left[iq^2-i\lambda r\int_0^1 dxV(\mathbf{r}_x)\right]\times \tag{21}$$

$$
\times \left\{ 1 - \frac{ir}{2\kappa} \left[ i\lambda r \int\limits_0^1 dx\, x \left(1 - \frac{r}{a_1} x\right) \Delta_{1\perp} V(\mathbf{r}_x) + \right.\right.
$$

$$
\left.\left. + 2r^2 \int\limits_0^1 dx \left(1 - \frac{r}{a_1} x\right) (\boldsymbol{\nabla}_{1\perp} V(\mathbf{r}_x)) \int\limits_0^x dy\, y \boldsymbol{\nabla}_{1\perp} V(\mathbf{r}_y) \right] \right\} ,
$$

where $\mathbf{r}_x = \mathbf{r}_1 + x\mathbf{r} + \mathbf{q}x\sqrt{2r(r-a_1)/(\kappa a_1)}$, $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. Let us remind, that the derivatives over $\mathbf{r}_1$ in this formula must be calculated at fixed $\mathbf{r}$.

Note that the term proportional to $\mathbf{q}$ in $\mathbf{r}_x$ is essential only in a narrow region $|x - a_1/r| \sim \rho_*/r \ll 1$. Using this fact, we can eliminate the quantity $a_1$ from the formula (21). In order to do this, we present $\mathbf{r}_x = \mathbf{R}_x + \delta\mathbf{r}_x$, where

$$
\mathbf{R}_x = \mathbf{r}_1 + x\mathbf{r} + \sqrt{2x(1-x)r/\kappa}\,\mathbf{q}, \quad \delta\mathbf{r}_x = \sqrt{\frac{2r}{\kappa}} \left( x\sqrt{r/a_1 - 1} - \sqrt{x(1-x)} \right) \mathbf{q} \,. \tag{22}
$$

expand $V(\mathbf{r}_x)$ in the exponent with respect to $\delta\mathbf{r}_x$ up to the first term, and replace $\mathbf{r}_x$ with $\mathbf{R}_x$ in the rest of the expression. After the integration by parts over $\mathbf{q}$ we obtain:

$$
\begin{aligned}
D^{(0)}(\mathbf{r}_2, \mathbf{r}_1|\,\varepsilon) \;&=\; \frac{i e^{i\kappa r}}{4\pi^2 r} \int d\mathbf{q} \exp\left[ iq^2 - i\lambda r \int_0^1 dx V(\mathbf{R}_x) \right] \times \\
&\times \left\{ 1 + \frac{ir^3}{2\kappa} \int\limits_0^1 dx \int\limits_0^x dy (x-y) (\boldsymbol{\nabla}_{1\perp} V(\mathbf{R}_x)) (\boldsymbol{\nabla}_{1\perp} V(\mathbf{R}_y)) \right\} .
\end{aligned} \tag{23}
$$

As mentioned above, the regions of applicability of two formulas (12) and (23) are overlapping. At $\rho_* \ll \min(a_1, a_2) \ll \kappa\rho_*^2$ it is possible to expand $V(\mathbf{R}_x)$ in $\mathbf{q}$ up to the second term and integrate over $\mathbf{q}$. After that the formula (23) turns into (12) including the terms of the order of $1/\kappa\rho_*$. Therefore, the result can be presented in the form which is correct everywhere for $\kappa\rho_* \gg 1$:

$$
\begin{aligned}
&D^{(0)}(\mathbf{r}_2, \mathbf{r}_1|\,\varepsilon) = \frac{i e^{i\kappa r}}{4\pi^2 r} \int d\mathbf{q} \exp\left[ iq^2 - i\lambda r \int_0^1 dx V(\mathbf{R}_x) \right] \times \\
&\times \left\{ 1 - \frac{\lambda}{2\kappa} \left[ 2 \int\limits_0^1 dx V(\mathbf{R}_x) - V(\mathbf{r}_1) - V(\mathbf{r}_2) \right] + \right. \\
&\left. + \frac{ir^3}{\kappa} \int\limits_0^1 dx \int\limits_0^x dy \left[ \sqrt{x(1-x)y(1-y)} - (1-x)y \right] (\boldsymbol{\nabla}_{1\perp} V(\mathbf{R}_x)) (\boldsymbol{\nabla}_{1\perp} V(\mathbf{R}_y)) \right\} .
\end{aligned} \tag{24}
$$

$\mathbf{R}_x$ is defined in (22). Indeed, in the first region we can expand $V(\mathbf{R}_x)$ in $\mathbf{q}$ up to the second-order terms and to integrate over $\mathbf{q}$, which leads to the formula (12). In the second region one should bear in mind, that the main contribution to integrals comes from a narrow region of $x$ and $y$ close to $a_1/r$ with $\delta x \sim \delta y \sim \rho_*/r$. Therefore

$$
2\left[ \sqrt{x(1-x)y(1-y)} - (1-x)y \right] = x - y - (x-y)^2 - \left( \sqrt{x(1-x)} - \sqrt{y(1-y)} \right)^2 \approx x - y
$$

up to the second-order terms in $\rho_*/r$. Besides, in this region the terms linear in $V$ can be neglected in (24). Thus, in the second region (24) turns into (23).

Using (2), we obtain the final expression for the Green function $D(\mathbf{r}_2, \mathbf{r}_1|\varepsilon)$

$$D(\mathbf{r}_2, \mathbf{r}_1|\,\varepsilon) = \frac{i\mathrm{e}^{i\kappa r}}{4\pi^2 r}\int d\mathbf{q}\exp\left[iq^2 - i\lambda r\int_0^1 dxV(\mathbf{R}_x)\right]\times \tag{25}$$
$$\times\left\{1 - \frac{r}{2\kappa}\int\limits_0^1 dx\,\boldsymbol{\alpha}\,\boldsymbol{\nabla}_1 V(\mathbf{R}_x) - \frac{\lambda}{2\kappa}\left[2\int\limits_0^1 dxV(\mathbf{R}_x) - V(\mathbf{r}_1) - V(\mathbf{r}_2)\right] + \right.$$
$$\left. + \frac{ir^3}{\kappa}\int\limits_0^1 dx\int\limits_0^x dy\left[\sqrt{x(1-x)y(1-y)} - (1-x)y\right](\boldsymbol{\nabla}_{1\perp}V(\mathbf{R}_x))(\boldsymbol{\nabla}_{1\perp}V(\mathbf{R}_y))\right\}\,.$$

The advantage of the representations (24) and (25) is that they keep their form in any reference frame.

The integration over the variable $\mathbf{q}$ in formulas (24) and (25) can be interpreted as the account for quantum fluctuations near the rectilinear trajectory between vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. The integral over $\mathbf{q}$ converges at $q \leq 1$. Using this fact, we conclude that the quantum fluctuations can be neglected, when

$$\int_0^1 dx\sqrt{x(1-x)r/\kappa}|\boldsymbol{\nabla}_{1\perp}V(\mathbf{r}_1 + x\mathbf{r})| \ll \int_0^1 dx|V(\mathbf{r}_1 + x\mathbf{r})|$$

This condition is actually equivalent to the first condition in (8) ensuring applicability of the eikonal approximation.

In the formulas (24) and (25) the terms containing a potential in the pre-exponent factor, give the correction to the quasiclassical Green function. The expressions obtained are valid when these correction are small. Our results were obtained for a localized potential. Nevertheless, they are valid for any potential if the correction is small, e.g. for a superposition of localized potentials.

## 2.1 Quasiclassical Green function in a central field

For a spherically symmetric potential the quasiclassical Green function without corrections can be obtained also from the results of [6, 7], where it was calculated for the case of almost collinear vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. In these papers with the help of quasiclassical radial wave functions the following expression for the function $D^{(0)}(\mathbf{r}_2, \mathbf{r}_1|\,\varepsilon)$ was obtained at small angle $\theta$ between vectors $\mathbf{r}_2$ and $-\mathbf{r}_1$:

$$D^{(0)}(\mathbf{r}_2, \mathbf{r}_1|\,\varepsilon) = \frac{i\mathrm{e}^{i\kappa(r_1+r_2)}}{4\pi\kappa r_1 r_2}\int\limits_0^\infty dl l J_0(l\theta)\exp\left\{i\left[\frac{l^2(r_1+r_2)}{2\kappa r_1 r_2} + 2\lambda\delta(l/\kappa) + \lambda(\Phi(r_1) + \Phi(r_2))\right]\right\}. \tag{26}$$

Here

$$\Phi(r) = \int\limits_r^\infty V(\zeta)d\zeta\,,\ \delta(\rho) = -\int\limits_0^\infty V\left(\sqrt{\zeta^2+\rho^2}\right)d\zeta\,,\ \lambda = \varepsilon/\kappa\,.$$

If the angle $\theta' = \pi - \theta$ between vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is small, then

$$D^{(0)}(\mathbf{r}_1, \mathbf{r}_2|\,\varepsilon) = -\frac{1}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|}\exp\{i\kappa|\mathbf{r}_1 - \mathbf{r}_2| + i\lambda\mathrm{sign}(r_1 - r_2)(\Phi(r_1) - \Phi(r_2))\}\,. \tag{27}$$

For further transformations it is convenient to rewrite the expression (26) in the other form, using the identity

$$\int dl\, l J_0(l\theta) g(l^2) = \frac{1}{2\pi}\int d\mathbf{q}\, \exp(i\mathbf{q}\,\boldsymbol{\theta})\, g(q^2)\,,$$

where $g(x)$ is an arbitrary function, and $\mathbf{q}$ is a two-dimensional vector. Let us substitute this relation to (26) and make the change of variables

$$\mathbf{q} \to \sqrt{\frac{2\kappa r_1 r_2}{r_1+r_2}}\mathbf{q} - \frac{\kappa r_1 r_2}{r_1+r_2}\boldsymbol{\theta}\,.$$

Defining an impact parameter $\boldsymbol{\rho}$ by a relation

$$\boldsymbol{\rho} = \frac{\mathbf{r}\times[\mathbf{r}_1\times\mathbf{r}_2]}{r^3}\,,$$

where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, taking into account, that at $\theta \ll 1$ the impact parameter $\boldsymbol{\rho} \approx \boldsymbol{\theta} r_1 r_2/(r_1 + r_2)$ and $\rho \ll r$, we can rewrite (26) as

$$D^{(0)}(\mathbf{r}_2, \mathbf{r}_1|\,\varepsilon) = \frac{i e^{i\kappa r}}{4\pi^2 r}\int d\mathbf{q}\exp\left[iq^2 - i\lambda r\int_0^1 dx V\left(\mathbf{r}_1 + x\mathbf{r} + \mathbf{q}\sqrt{2r_1r_2/\kappa r}\right)\right]\,. \qquad (28)$$

Here $\mathbf{q}$ is a two-dimensional vector perpendicular to vector $\mathbf{r}$. This formula was obtained for small angles $\theta$. However, it is correct also for $\theta \sim 1$, since in this case the term, proportional to $\mathbf{q}$ the in argument of a potential can be neglected and the Green function (28) turns into the eikonal one. In particular, for $\theta' \ll 1$ it coincides with (27). The expression (28) agrees with the main (without the correction) term of (24) since in the main approximation it is possible to replace $\sqrt{x(1-x)}$ by $\sqrt{r_1 r_2/r^2}$.

# 3 Wave functions in the quasiclassical approximation

The obtained expressions for the quasiclassical Green function allow us to find the quasiclassical wave functions with the first corrections. The quasiclassical wave functions were found earlier in papers [8, 9] for the case of Coulomb field (Furry-Sommerfeld-Maue wave function) and in [10] for arbitrary decreasing central potential. These wave functions were calculated in the main approximation. In paper [11] the wave functions and corrections to them were found for arbitrary potential in the eikonal approximation.

To calculate the wave functions, we use the known (see, for example, [12]) relations

$$\lim_{r_2\to\infty} D^{(0)}(\mathbf{r}_2, \mathbf{r}_1|\varepsilon) = -\frac{\exp[i\kappa r_2]}{4\pi r_2}\psi^{(-)*}_{\mathbf{p}_2}(\mathbf{r}_1)\,, \qquad (29)$$
$$\lim_{r_1\to\infty} D^{(0)}(\mathbf{r}_2, \mathbf{r}_1|\varepsilon) = -\frac{\exp[i\kappa r_1]}{4\pi r_1}\psi^{(+)}_{\mathbf{p}_1}(\mathbf{r}_2)\,.$$

Here $\mathbf{p}_1 = -\kappa\mathbf{r}_1/r_1$, $\mathbf{p}_2 = \kappa\mathbf{r}_2/r_2$, $\psi^{(+)}_{\mathbf{p}}(\mathbf{r})$ ( $\psi^{(-)}_{\mathbf{p}}(\mathbf{r})$) denotes a solution of the Klein-Gordon equation containing at the infinity a plane wave with momentum $\mathbf{p}$ and a diverging

(converging) spherical wave. From (24) and (29) we obtain for the quasiclassical wave function of the Klein-Gordon equation with the first correction taken into account

$$
\begin{aligned}
&\psi_{\mathbf{p}}^{(\pm)}(\mathbf{r}) = \pm\int\frac{d\mathbf{q}}{i\pi}\exp\left[i\mathbf{p}\mathbf{r}\pm iq^2\mp i\lambda\int_0^\infty dxV(\mathbf{r}_x)\right]\times \\
&\times\left\{1+\frac{\lambda}{2\kappa}V(\mathbf{r})\pm\frac{i}{\kappa}\int\limits_0^\infty dx\int\limits_0^x dy\,[\sqrt{xy}-y]\,(\boldsymbol{\nabla}_\perp V(\mathbf{r}_x))(\boldsymbol{\nabla}_\perp V(\mathbf{r}_y))\right\}\ , \\
&\mathbf{r}_x=\mathbf{r}\mp\mathbf{p}x/\kappa+\mathbf{q}\sqrt{2x/\kappa}\ ,\quad \kappa=|\mathbf{p}|\ .
\end{aligned}
\tag{30}
$$

Here $\mathbf{q}$ is a two-dimensional vector perpendicular to $\mathbf{p}$, $\boldsymbol{\nabla}_\perp$ is the component of a gradient, perpendicular to $\mathbf{p}$. Similarly, for the Dirac equation the quasiclassical wave function with the first correction is obtained from (25):

$$
\begin{aligned}
&\Psi_{\mathbf{p}}^{(\pm)}(\mathbf{r}) = \pm\int\frac{d\mathbf{q}}{i\pi}\exp\left[i\mathbf{p}\mathbf{r}\pm iq^2\mp i\lambda\int_0^\infty dxV(\mathbf{r}_x)\right]\left\{1\mp\frac{1}{2\kappa}\int\limits_0^\infty dx\,\boldsymbol{\alpha}\ \boldsymbol{\nabla}V(\mathbf{r}_x)+\right. \\
&\left.+\frac{\lambda}{2\kappa}V(\mathbf{r})\pm\frac{i}{\kappa}\int\limits_0^\infty dx\int\limits_0^x dy\,(\sqrt{xy}-y)\,(\boldsymbol{\nabla}_\perp V(\mathbf{r}_x))(\boldsymbol{\nabla}_\perp V(\mathbf{r}_y))\right\}u_{\mathbf{p}}^{\lambda}\ ,
\end{aligned}
\tag{31}
$$

where $u_{\mathbf{p}}^{\lambda}$ is a free positive-energy ($\lambda=1$) and negative-energy ($\lambda=-1$) Dirac spinors with momentum $\mathbf{p}$. In the case, when $V(\mathbf{r}_x)$ can be expanded in $\mathbf{q}$, the formula (31) turns into the wave function in the eikonal approximation with correction being in accordance with the result of [11].

# 4 Scattering of charged particles

In this section we use the obtained quasiclassical Green function for the calculation of the small-angle scattering amplitude of the high-energy charged particle in localized potential. In the case of a particle with spin zero the scattering amplitude has the form:

$$
f(\mathbf{p}_2,\mathbf{p}_1) = -\frac{\kappa}{2\pi}\int d\mathbf{r}\psi_{\mathbf{p}_2}^{(-)*}(\mathbf{r})\phi(\mathbf{r})\mathrm{e}^{i\mathbf{p}_1\mathbf{r}}\ . \tag{32}
$$

Recollect, that $\phi=\lambda V-V^2/2\kappa$.

For the sake convenience we put the origin of the reference frame at the point of potential maximum and pass to cylindrical coordinates with axis $z$ along the vector $\mathbf{p}_2$. Substituting (30) in (32), we obtain with the account of the first correction

$$
\begin{aligned}
f &= \frac{i\kappa}{2\pi^2}\int\limits_{-\infty}^{\infty}dz\int d\boldsymbol{\rho}\int d\mathbf{q}\exp\left[-iQ_z z-i\mathbf{Q}_\perp\boldsymbol{\rho}+i\mathbf{q}^2-i\lambda\int_0^\infty dxV(x+z,\boldsymbol{\rho}_x)\right]\times \\
&\times\lambda V(z,\boldsymbol{\rho})\left[1+\frac{i}{\kappa}\int\limits_0^\infty dx\int\limits_0^x dy\,[\sqrt{xy}-y]\,(\boldsymbol{\nabla}_\rho V(x+z,\boldsymbol{\rho}_x))(\boldsymbol{\nabla}_\rho V(y+z,\boldsymbol{\rho}_y))\right]\ ,
\end{aligned}
\tag{33}
$$

where $\boldsymbol{\rho}_x=\boldsymbol{\rho}+\sqrt{2x/\kappa}\,\mathbf{q}$, $\mathbf{Q}=\mathbf{p}_2-\mathbf{p}_1$.

We demonstrate first that in the scattering problem not the eikonal wave function but the quasiclassical one should be used. For this purpose we calculate the amplitude in the main approximation, that corresponds to the replacement of the factor in square brackets of (33) by 1. We split the region of integration over $z$ into two: $(-\infty, 0)$ and $(0, \infty)$. In the region from zero to infinity it is possible to neglect quantum fluctuations, which means the replacement $\boldsymbol{\rho}_x \to \boldsymbol{\rho}$ and then to integrate over $\mathbf{q}$. Finally, the contribution of this region reads

$$f_+ = -\frac{\kappa}{2\pi}\int\limits_{-\infty}^{\infty} dz \int d\boldsymbol{\rho} \exp\left[-iQ_z z - i\mathbf{Q}_\perp\boldsymbol{\rho} - i\lambda\int_0^\infty dx V(x+z,\boldsymbol{\rho})\right]\lambda V(z,\boldsymbol{\rho})\ . \tag{34}$$

Now, integrating by parts over $z$ with the help of the relation

$$\lambda V(z,\boldsymbol{\rho})\exp\left[-i\lambda\int_0^\infty dx V(x+z,\boldsymbol{\rho})\right] = -i\frac{\partial}{\partial z}\left\{\exp\left[-i\lambda\int_0^\infty dx V(x+z,\boldsymbol{\rho})\right]-1\right\}\ , \tag{35}$$

we obtain

$$\begin{aligned} f_+ \ &= \ -\frac{i\kappa}{2\pi}\int d\boldsymbol{\rho}\exp[-i\mathbf{Q}_\perp\boldsymbol{\rho}]\Bigg(\exp\left[-i\lambda\int_0^\infty dx V(x,\boldsymbol{\rho})\right]-1- \\ &\quad -iQ_z\int\limits_0^\infty dz\ \exp[-iQ_z z]\left(\exp\left[-i\lambda\int_0^\infty dx V(x,\boldsymbol{\rho})\right]-1\right)\Bigg)\ . \end{aligned} \tag{36}$$

The main contribution to the integral over $z$ in this formula comes from $z \sim \rho$. The relative magnitude of the contribution proportional to this integral, in comparison with the first term is $Q_z\rho \sim Q_z/Q_\perp \ll 1$, therefore in the main approximation this contribution can be neglected. In the region from $-\infty$ up to zero in the main approximation we can replace $x$ with $|z|$ in $\boldsymbol{\rho}_x$. Performing the shift $\boldsymbol{\rho} \to \boldsymbol{\rho} - \mathbf{q}\sqrt{2|z|/\kappa}$ we obtain

$$\begin{aligned} f_- \ &= \ \frac{i\kappa}{2\pi^2}\int\limits_{-\infty}^{0} dz\int d\boldsymbol{\rho}\int d\mathbf{q}\exp\Big[-i(Q_z - \mathbf{Q}_\perp^2/2\kappa)\,z - i\mathbf{Q}_\perp\boldsymbol{\rho} + i(\mathbf{q}+\mathbf{Q}_\perp\sqrt{|z|/2\kappa})^2 - \\ &\quad -i\lambda\int_0^\infty dx V(x+z,\boldsymbol{\rho})\Big]\,\lambda V(z,\boldsymbol{\rho}-\mathbf{q}\sqrt{2|z|/\kappa})\ . \end{aligned} \tag{37}$$

Note, that at $\kappa\rho \gg 1$ at any $z$ the condition $\sqrt{2|z|/\kappa} \ll \max(|z|,\rho)$ holds, which allows us to neglect the term proportional to $\mathbf{q}$ in the argument of the potential. In the small-angle approximation $Q_z = \mathbf{Q}_\perp^2/2\kappa$, that is, the term $\propto z$ in the exponent vanishes. Taking the integral over $\mathbf{q}$, and then over $z$ with the use of the identity (35), we find

$$f_- = -\frac{i\kappa}{2\pi}\int d\boldsymbol{\rho}\exp[-i\mathbf{Q}_\perp\boldsymbol{\rho}]\left(\exp\left[-i\lambda\int_{-\infty}^\infty dx V(x,\boldsymbol{\rho})\right] - \exp\left[-i\lambda\int_0^\infty dx V(x,\boldsymbol{\rho})\right]\right)\ . \tag{38}$$

Combining $f_+$ and $f_-$, we obtain the expression known as the scattering amplitude in the eikonal approximation:

$$f = -\frac{i\kappa}{2\pi}\int d\boldsymbol{\rho}\exp[-i\mathbf{Q}_\perp\boldsymbol{\rho}]\left(\exp\left[-i\lambda\int_{-\infty}^\infty dx V(x,\boldsymbol{\rho})\right]-1\right)\ . \tag{39}$$

Usually this result is derived with the use of the eikonal wave function in the whole region of integration over $z$ and neglecting the term $Q_z z$ in the exponent. It follows from our

consideration that at arbitrary momentum transfers both these approximations, generally speaking, are incorrect. For example, consider a screened Coulomb potential with the radius of screening $r_c$. The main contribution to the amplitude comes from the region $z \sim r_c$, $\rho \sim 1/Q_\perp$. If the quantity $Q_z z \sim Q_\perp^2 r_c/\kappa$ is not small in this region as compared to unity, the term with $\mathbf{q}$ in $\boldsymbol{\rho}_x$ can not be neglected, since $\rho \sim 1/Q_\perp \leq \sqrt{r_c/\kappa}$. Thus, the eikonal wave function becomes inapplicable in the region of the main contribution to the amplitude. Additionally, the term $Q_z z$ in the exponent can not be neglected. Keeping this term and still using the eikonal wave function leads under condition of $Q_\perp^2 r_c/\kappa \geq 1$ to wrong result for the scattering amplitude. In particular, acting so, one can not reproduce a well known result(Ratherford formula) in the limit $r_c \to \infty$ (unscreened Coulomb potential). Thus, the formula (39) is valid for any $Q_\perp \ll \kappa$, however, its correct derivation can not be done with the use of the eikonal wave function.

Let us pass now to the calculation of scattering amplitude with the first correction. For the sake convenience we first set the lower limit of integration over $z$ to $-L$ , and then take the limit $L \to \infty$. Using the identity

$$\left[\lambda V(z,\boldsymbol{\rho}) + i\frac{\partial}{\partial z} + \lambda \int\limits_0^\infty dx \frac{\mathbf{q}}{\sqrt{2\kappa x}} \boldsymbol{\nabla}_\rho V(z+x,\rho_x)\right] \exp\left[-i\lambda \int_0^\infty dx V(x+z,\boldsymbol{\rho}_x)\right] = 0 \ , \tag{40}$$

it is possible to perform the integration by parts over $z$ in the first term of square brackets of (33). As a result, the expression for $f$ acquires the form $f = f_0 + f_1$, where

$$f_0 = -\frac{\kappa}{2\pi^2} \lim_{L\to\infty} \int d\boldsymbol{\rho} \int d\mathbf{q} \exp\left[i\frac{\mathbf{Q}_\perp^2}{2\kappa}L - i\mathbf{Q}_\perp\boldsymbol{\rho} + i\mathbf{q}^2 - i\lambda \int_{-L}^\infty dx V(x,\boldsymbol{\rho}_{x+L})\right] \ , \tag{41}$$

$$\begin{aligned} f_1 = {} & \frac{i\kappa}{2\pi^2} \int\limits_{-\infty}^\infty dz \int d\boldsymbol{\rho} \int d\mathbf{q} \exp\left[-i\frac{\mathbf{Q}_\perp^2}{2\kappa}z - i\mathbf{Q}_\perp\boldsymbol{\rho} + i\mathbf{q}^2 - i\lambda \int_0^\infty dx V(x+z,\boldsymbol{\rho}_x)\right] \times \\ & \times \left[\frac{i\lambda}{\kappa} V(z,\boldsymbol{\rho}) \int\limits_0^\infty dx \int\limits_0^x dy \left(\sqrt{xy} - y\right) (\boldsymbol{\nabla}_\rho V(x+z,\boldsymbol{\rho}_x))(\boldsymbol{\nabla}_\rho V(y+z,\boldsymbol{\rho}_y)) + \right. \\ & \left. + \frac{\mathbf{Q}_\perp^2}{2\kappa} - \lambda \int\limits_0^\infty dx \frac{\mathbf{q}}{\sqrt{2\kappa x}} \boldsymbol{\nabla}_\rho V(z+x,\rho_x)\right] \ . \end{aligned} \tag{42}$$

The term independent of potential and vanishing at $\mathbf{Q}_\perp \neq 0$, is omitted in expression for $f_0$. We will take it into account explicitly in the final expression for the amplitude. In order to find the limit $L \to \infty$ in function $f_0$, we make the shifts $\boldsymbol{\rho} \to \boldsymbol{\rho} - \mathbf{q}\sqrt{2L/\kappa}$ and $\mathbf{q} \to \mathbf{q} - \mathbf{Q}_\perp\sqrt{L/2\kappa}$. After that the calculation of this limit and integration over $\mathbf{q}$ become elementary. With the correction of the order of $Q_\perp/\kappa$ taken into account we obtain

$$f_0 = -\frac{i\kappa}{2\pi} \int d\boldsymbol{\rho} \exp\left[-i\mathbf{Q}_\perp\boldsymbol{\rho} - i\lambda \int_{-\infty}^\infty dx V(x,\boldsymbol{\rho})\right] \left[1 + \frac{i\lambda}{2\kappa} \int\limits_{-\infty}^\infty dx\, x \mathbf{Q}_\perp \boldsymbol{\nabla}_\rho V(x,\boldsymbol{\rho})\right] \tag{43}$$

The integral over $\boldsymbol{\rho}$ in the expression for $f_1$ (42) converges at $\rho \sim 1/Q_\perp$. The quantum fluctuations are important only at $z < 0$ and $x$ close to $-z$. Since the quantity $f_1$ is small being a part of the correction, $\boldsymbol{\rho}_x$ in the formula (42) can be replaced by $\boldsymbol{\rho}_{|z|}$. Besides the factor $V(z,\boldsymbol{\rho})$ can be replaced by $V(z,\rho_{|z|})$ because the difference between these two is

small at any $z$. The integral over $z$ from zero to infinity converges at $z \le \rho$, so $z\mathbf{Q}_\perp^2/2\kappa \sim \rho\mathbf{Q}_\perp^2/\kappa \sim Q_\perp/\kappa \ll 1$. Therefore, it is possible to replace in the exponent $z\mathbf{Q}_\perp^2/2\kappa$ by $-|z|\mathbf{Q}_\perp^2/2\kappa$. After integrating by parts over $\mathbf{q}$ the term proportional to $\mathbf{q}$ this vector enters everywhere, except the exponent, only in the argument of potential. Then the expression for $f_1$ can be transformed to the form

$$f_1 = \int\limits_{-\infty}^{\infty} dz \int d\boldsymbol{\rho} \int \frac{d\mathbf{q}}{i\pi} \exp\left[i\frac{\mathbf{Q}_\perp^2}{2\kappa}|z| - i\mathbf{Q}_\perp\boldsymbol{\rho} + i\mathbf{q}^2\right] g(z, \boldsymbol{\rho} + \sqrt{2|z|/\kappa}\,\mathbf{q}) \tag{44}$$

with some function $g$. If we perform the shift $\boldsymbol{\rho} \to \boldsymbol{\rho} - \mathbf{q}\sqrt{2|z|/\kappa}$ the integral over $\mathbf{q}$ becomes elementary. As a result we have

$$f_1 = \int\limits_{-\infty}^{\infty} dz \int d\boldsymbol{\rho} \exp\left[-i\mathbf{Q}_\perp\boldsymbol{\rho}\right] g(z, \boldsymbol{\rho}) \ . \tag{45}$$

Finally, we obtain the following expression for $f_1$

$$\begin{aligned} f_1 \;=\; & -\frac{1}{2\pi}\int d\boldsymbol{\rho} \int\limits_{-\infty}^{\infty} dz \exp\left[-i\mathbf{Q}_\perp\boldsymbol{\rho} - i\lambda\int_0^\infty dx V(x+z,\boldsymbol{\rho})\right] \times \\ & \times \Bigg[ i\lambda V(z,\boldsymbol{\rho}) \int\limits_0^\infty dx \int\limits_0^x dy\, (\sqrt{xy} - y)\, (\boldsymbol{\nabla}_\rho V(x+z,\boldsymbol{\rho}))(\boldsymbol{\nabla}_\rho V(y+z,\boldsymbol{\rho})) + \\ & + \frac{1}{2}\int\limits_0^\infty dx \int\limits_0^\infty dy \left(1 - \sqrt{x/y}\right) (\boldsymbol{\nabla}_\rho V(x+z,\boldsymbol{\rho}))(\boldsymbol{\nabla}_\rho V(y+z,\boldsymbol{\rho}))\Bigg] \end{aligned} \tag{46}$$

When deriving this formula we have integrated by parts over $\boldsymbol{\rho}$ the term proportional to $\mathbf{Q}_\perp^2$. As can be checked, the integrand in (46) is a total derivative over $z$ and the integration over $z$ becomes trivial. Combining obtained expression with (43), and integrating by parts over $\boldsymbol{\rho}$ the term proportional to $\mathbf{Q}_\perp$, we obtain the scattering amplitude with the first correction

$$\begin{aligned} f \;=\; & -\frac{i\kappa}{2\pi}\int d\boldsymbol{\rho} \exp\left[-i\mathbf{Q}_\perp\boldsymbol{\rho}\right] \Bigg\{ \exp\left[-i\lambda\int_{-\infty}^\infty dx V(x,\boldsymbol{\rho})\right] - 1 + \\ & + \exp\left[-i\lambda\int_{-\infty}^\infty dx V(x,\boldsymbol{\rho})\right] \Bigg[ \frac{\lambda}{2\kappa}\int\limits_{-\infty}^{\infty} dx\, x\Delta_\rho V(x,\boldsymbol{\rho}) - \\ & - \frac{i}{\kappa}\int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{x} dy y (\boldsymbol{\nabla}_\rho V(x,\boldsymbol{\rho}))(\boldsymbol{\nabla}_\rho V(y,\boldsymbol{\rho}))\Bigg]\Bigg\} \ . \end{aligned} \tag{47}$$

Here $\Delta_\rho = \boldsymbol{\nabla}_\rho^2$. Using the wave function (31), it is possible to show (see, for example, [13]), that the small-angle scattering amplitude for particles with spin 1/2 coincides with amplitude (47) for spin-zero particles including the terms of the order of $Q_\perp/\kappa$.

The obtained amplitude has correct properties with respect to shifts. It follows from (33), that after the replacement $V(\mathbf{r}) \to V(\mathbf{r}+\mathbf{r}_0)$ the amplitude $f(\mathbf{p}_2, \mathbf{p}_1)$ acquires the factor $\exp[i\mathbf{Q}\mathbf{r}_0]$. The amplitude (47) obviously has this property with respect to the shift in the direction, perpendicular to $\mathbf{p}_2$, i.e., to $z$-axis. Let us consider now the shift along

the $z$-axis: $V(z,\boldsymbol{\rho}) \to V(z+z_0,\boldsymbol{\rho})$. Performing the change of variables $x \to x-z_0$ and $y \to y-z_0$ in the integrals over $x$ and $y$ , we obtain the additional term proportional to $z_0$:

$$
\begin{aligned}
\delta f &= \frac{iz_0}{4\pi}\int d\boldsymbol{\rho}\exp\left[-i\mathbf{Q}_\perp\boldsymbol{\rho} - i\lambda\int_{-\infty}^{\infty}dxV(x,\boldsymbol{\rho})\right]\times \\
&\times\left[\lambda\int\limits_{-\infty}^{\infty}dx\,\Delta_\rho V(x,\boldsymbol{\rho}) - i\left(\int\limits_{-\infty}^{\infty}dx\boldsymbol{\nabla}_\rho V(x,\boldsymbol{\rho})\right)^2)\right] \\
&= -\frac{z_0}{4\pi}\int d\boldsymbol{\rho}\exp\left[-i\mathbf{Q}_\perp\boldsymbol{\rho}\right]\Delta_\rho\left(\exp[-i\lambda\int_{-\infty}^{\infty}dxV(x,\boldsymbol{\rho})\right]-1\right)\ .
\end{aligned} \tag{48}
$$

Integrating by parts over $\boldsymbol{\rho}$ and substituting $\mathbf{Q}_\perp^2/2\kappa \to Q_z$, we find for $Q_zZ_0 \ll 1$ that $f+\delta f \approx f\exp(iQ_zz_0)$. Thus, the correct transformation properties of the scattering amplitude with respect to shifts holds with the same accuracy, as formula (47) itself.

For the potential satisfying the condition $V(z,\boldsymbol{\rho}) = V(-z,\boldsymbol{\rho})$ the expression (47) agrees with that obtained in [11] in the eikonal approximation with the account for the correction. However, as explained above, the formula (47) holds even when the eikonal approximation is inapplicable.

# 5 Delbruck scattering

The process of coherent photon scattering in the electric field of atoms via virtual electron -positron pairs (Delbruck scattering) has been intensively investigated both theoretically, and experimentally (see, e.g., the review [14]). In this section we consider the forward Delbruck scattering in a screened Coulomb field as one more example of the application of the quasiclassical Green function obtained. The photon energy $\omega$ is assumed to be large in comparison with the electron mass $m$. According to the optical theorem, the imaginary part of this amplitude is proportional to the cross section of electron-positron pair production by a photon in the field of atom. To describe the propagation of light in matter it is also necessary to know the real part of this amplitude.

For the calculation of amplitudes of Delbruck scattering with the help of Green functions one is forced to use different approximations for these functions for different momentum transfers $\boldsymbol{\Delta} = \mathbf{k}_2 - \mathbf{k}_1$ ($\mathbf{k}_1, \mathbf{k}_2$ are the momenta of initial and final photons). At $\Delta \sim \omega$ the quasiclassical approximation is inapplicable, since the basic contribution to the amplitude is given by the orbital moment $l \sim \omega/\Delta \sim 1$. For such momentum transfers the amplitude was calculated in [15]. In region $\omega \gg \Delta \gg m^2/\omega$ it is possible to use the quasiclassical Green function in the main approximation [1, 16]. The calculation of Delbruck scattering amplitude in Coulomb field at $\Delta \leq m^2/\omega$ required a special consideration (see [17, 2]). In this case the contribution to the amplitude is given by the impact parameters $\rho$ up to $\rho \sim \omega/m^2$. At such impact parameters it is necessary to take into account a correction to the quasiclassical Green function [2]. For the screened Coulomb potential with $\omega/m^2 \gg r_c$ ( $r_c$ is the radius of screening, in Thomas-Fermi model $r_c \sim (M\alpha)^{-1}Z^{-1/3} \gg 1/m$) the contribution to the amplitude is given by the impact parameters $\rho \leq r_c \ll \omega/m^2$. In this case, the quasiclassical Green function without corrections can be used for arbitrary $\Delta \ll \omega$ [6, 7]. However, if $\Delta$, $r_c^{-1} \leq m^2/\omega$, the correction is very important. Moreover,

the expression for the forward scattering amplitude ($\Delta = 0$), obtained with the use of the quasiclassical Green function without corrections is, strictly speaking, indefinite. Below we derive the amplitude of forward Delbruck scattering at arbitrary ratio between $r_c$ and $\omega/m^2$.

As shown in [6], the amplitude of forward Delbruck scattering for high photon energy can be presented as follows

$$
\begin{aligned}
M &= i\alpha \int\limits_0^\omega d\varepsilon \int d\mathbf{r}_1 d\mathbf{r}_2 \exp[i\mathbf{k}(\mathbf{r}_1 - \mathbf{r}_2)] \times \\
&\times \mathrm{Sp}\Big[(2\mathbf{e}^*\mathbf{p}_2 - \hat{e}^*\hat{k})D(\mathbf{r}_2, \mathbf{r}_1|\omega - \varepsilon)\Big]\Big[(2\mathbf{e}\mathbf{p}_1 + \hat{e}\hat{k})D(\mathbf{r}_1, \mathbf{r}_2| - \varepsilon)\Big]
\end{aligned} \tag{49}
$$

where $e$, $k$ are polarization and 4-momentum vectors of a photon, $\mathbf{p}_{1,2} = -i\boldsymbol{\nabla}_{1,2}$. In this formula the subtraction from integrand of its value at zero external field is assumed to be done. Since in the case of central field the amplitude of forward scattering does not depend on polarization of photon, it is convenient to make the substitution $e_i^* e_j \to (\delta_{ij} - k_i k_j/\omega^2)/2$ in (49).

Let us pass in (49) from variables $\mathbf{r}_{1,2}$ to

$$
\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \quad \boldsymbol{\rho} = \frac{\mathbf{r} \times [\mathbf{r}_1 \times \mathbf{r}_2]}{r^3}, \quad z = -\frac{(\mathbf{r}\mathbf{r}_1)}{r^2}\,.
$$

Since $\boldsymbol{\rho}\mathbf{r} = 0$, the integration over $\boldsymbol{\rho}$ is carried out in a plane, perpendicular to $\mathbf{r}$. The main contribution to the amplitude comes from the integration region, where $r \sim \omega/m^2$, $|z| \sim 1$ and angles between vector $\mathbf{r}$ and $\mathbf{k}$ are of the order $\theta_r \sim m/\omega \ll 1$. Due to the smallness of angles $\theta_r$ it is possible to consider vector $\boldsymbol{\rho}$ to be perpendicular to $\mathbf{k}$ too. Besides, it is obvious that the main contribution is given by $\rho \leq \min(1,\, m^2 r_c/\omega)$.

Let us split the region of integration over $\rho$ into two: from 0 up to $\rho_0$ and from $\rho_0$ to $\infty$, where $m/\omega \ll \rho_0 \ll \min(1,\, m^2 r_c/\omega)$. In the first region (at $\rho < \rho_0$) the following form of the quasiclassical Green function can be used

$$
\begin{aligned}
D(\mathbf{r}_2, \mathbf{r}_1|\,\varepsilon) &= \frac{i e^{i\kappa r}}{4\pi^2 r} \int d\mathbf{q} \left[1 + \frac{\boldsymbol{\alpha}\,\mathbf{q}}{\varepsilon}\sqrt{\frac{\kappa r}{2 r_1 r_2}}\right] \times \\
&\times \exp\left[i q^2 - i\lambda r \int_0^1 dx V\left(\mathbf{r}_1 + x\mathbf{r} + \mathbf{q}\sqrt{2 r_1 r_2/\kappa r}\right)\right],
\end{aligned} \tag{50}
$$

which we obtained by substituting (28) in (2) and performing some transformations of the term containing $\boldsymbol{\alpha}$ -matrix. Namely, the longitudinal components of gradient, which in comparison with transverse ones have additional smallness $\rho$, were omitted and the integration by parts over $\mathbf{q}$ was performed. Besides, by virtue of the definition of $\rho_0$, in this region we can neglect screening and replace $V(r)$ by a Coulomb potential $V_c(r) = -Z\alpha/r$. After this the integral over $x$ in the exponent can be easily taken.

The screening is essential only in the second region, where we can use the representation (13), that is, the eikonal Green function with the first correction. The substitution of (13) into (49) results in the cancellation of the potential-dependent terms in the exponent.

Therefore, the contribution from the second region does not contain any powers of the potential except the second one, which corresponds to the first Born approximation. Besides, due to the phase cancellation we must take into account the correction to the Green function. In this region the contributions to the amplitude rising from the correction and from the main term in Green function turn out to be of the same order of magnitude.

Going over to the calculation of the contribution $M_1$ from the first region, we substitute the Green function (50) for a pure Coulomb potential into (49) , differentiate and take the trace. Using the smallness of angles between the vectors $\mathbf{r}$ and $\mathbf{k}$ and expanding $\sqrt{\varepsilon^2 - m^2} \approx |\varepsilon| - m^2/2|\varepsilon|$, we obtain

$$
\begin{aligned}
M_1 \;=\; & -\frac{i\alpha}{(2\pi)^4}\int\limits_0^\omega d\varepsilon\varepsilon\kappa\int dr\, r^5 d\boldsymbol{\theta}_r \int\limits_{\rho<\rho_0} d\boldsymbol{\rho}\int\limits_0^1 \frac{dz}{(z(1-z))^3}\int d\mathbf{q}_1 d\mathbf{q}_2\,\times \\
& \times\left[\mathrm{Re}\left(\frac{|\boldsymbol{\rho}-\mathbf{q}_1|}{|\boldsymbol{\rho}-\mathbf{q}_2|}\right)^{2iZ\alpha} - 1\right]\exp\left\{i\frac{r}{2}\left[\omega\boldsymbol{\theta}_r^2 - \frac{m^2\omega}{\varepsilon\kappa} + \frac{\varepsilon q_1^2 + \kappa q_2^2}{z(1-z)}\right]\right\}\times \\
& \times\left[2\varepsilon\kappa[\mathbf{q}_1\mathbf{q}_2 - z(1-z)\boldsymbol{\theta}_r^2] + \frac{\omega}{4(z(1-z))}(\varepsilon\mathbf{q}_1 - \kappa\mathbf{q}_2,\, \mathbf{q}_1 - \mathbf{q}_2) - i\frac{\omega}{r}\right],
\end{aligned}
\tag{51}
$$

where $\kappa = \omega - \varepsilon$ as well as vectors $\mathbf{q}_{1,2}$, and $\boldsymbol{\theta}_r$ are two-dimensional vectors, perpendicular to $\mathbf{k}$. Note that the integral over $z$ in this formula is taken in the limits from zero to unity. The reason is that outside this interval the Green function have the eikonal form and the phases depending on a potential are cancelled. For $\rho < \rho_0 \ll 1$ it results in a negligible contribution from the region outside the interval $0 \le z \le 1$ compared to the contribution from this interval.

Let us integrate now over $\boldsymbol{\theta}_r$, pass from the variables $\mathbf{q}_{1,2}$ to $\mathbf{Q} = (\mathbf{q}_1 + \mathbf{q}_2)/2$ and $\mathbf{q} = (\mathbf{q}_1 - \mathbf{q}_2)/2$ and make the shift $\boldsymbol{\rho} \to \boldsymbol{\rho} + \mathbf{Q}$. After that the integral over $\boldsymbol{\rho}$ acquires the form

$$
J = \int\limits_{|\boldsymbol{\rho}+\mathbf{Q}|<\rho_0} d\boldsymbol{\rho}\left[\mathrm{Re}\left(\frac{|\boldsymbol{\rho}-\mathbf{q}|}{|\boldsymbol{\rho}+\mathbf{q}|}\right)^{2iZ\alpha} - 1\right]. \tag{52}
$$

The main contribution to the amplitude (51) comes from the region $Q,\, q \sim m/\omega \ll \rho_0$, where we can neglect $\mathbf{Q}$ in the limit of integration in (52). To take this integral (52) we subtract and add to the integrand the function $-2(Z\alpha)^2\,[2\boldsymbol{\rho}\mathbf{q}/(\rho^2+q^2)]^2$, which is easily integrated:

$$
J_1 = \int\limits_{\rho<\rho_0} d\boldsymbol{\rho}\left[-2(Z\alpha)^2\left(\frac{2\boldsymbol{\rho}\mathbf{q}}{\rho^2+q^2}\right)^2\right] = -4\pi(Z\alpha)^2 q^2\left(\ln\frac{\rho_0^2}{q^2} - 1\right). \tag{53}
$$

In turn, dealing with the difference we can extend the integral over $\rho$ to infinity owing to the fast convergence of the integral. To calculate this integral, it is convenient to multiply the integrand by

$$
1 \equiv \int_{-1}^{1} dy\,\delta\left(y - \frac{2\boldsymbol{\rho}\mathbf{q}}{\rho^2+q^2}\right) = (\rho^2+q^2)\int_{-1}^{1}\frac{dy}{|y|}\delta((\boldsymbol{\rho}-\mathbf{q}/y)^2 - q^2(1/y^2-1)), \tag{54}
$$

and change the order of integration over $\boldsymbol{\rho}$ and $y$. Integrating over $\boldsymbol{\rho}$, we obtain

$$
J_2 = 4\pi q^2\int\limits_0^1\frac{dy}{y^3}\left[\mathrm{Re}\left(\frac{1-y}{1+y}\right)^{iZ\alpha} - 1 + 2(Z\alpha)^2 y^2\right] \tag{55}
$$

Using the replacement $y = \tanh\tau$ we find for $J = J_1 + J_2$

$$J = 8\pi q^2 (Z\alpha)^2 \left[\ln\frac{2q}{\rho_0} - 1 + \mathrm{Re}\psi(1+iZ\alpha) + C\right] \tag{56}$$

where $C = 0.577...$ is the Euler constant, $\psi(x) = d\ln\Gamma(x)/dx$. It is convenient to take the remaining integrals in the following order: over $\mathbf{Q}$, $\mathbf{q}$, $r$, $z$, and $\varepsilon$. Finally, the contribution from the first region reads

$$M_1 = i\frac{28\alpha(Z\alpha)^2\omega}{9m^2}\left[\ln\frac{\omega\rho_0}{m} - i\frac{\pi}{2} - \mathrm{Re}\psi(1+iZ\alpha) - C - \frac{47}{42}\right]. \tag{57}$$

The contribution of the higher orders of the perturbation theory in the external field (Coulomb corrections) is given by the term $-\mathrm{Re}\psi(1+iZ\alpha) - C$ in (57) and coincides with the known result [17]. Thus, the Coulomb corrections are completely determined by the first region, when the quasiclassical Green function is not reduced to the eikonal one.

Let us pass to the calculation of the contribution $M_2$ from the second region. Taking the derivatives over $\mathbf{r}_{1,2}$, calculating the trace over gamma-matrices and integrating over $\boldsymbol{\theta}_r$ we come to the following representation for $M_2$:

$$\begin{aligned} M_2 &= \frac{\alpha}{2\pi\omega}\int\limits_0^\omega d\varepsilon\int dr\, r^2 \exp\left[-i\frac{\omega r m^2}{2\varepsilon\kappa}\right]\int\limits_{\rho>\rho_0} d\boldsymbol{\rho}\int\limits_{-\infty}^{\infty} dz \iint\limits_{0\,0}^{1\,1} dx\, dy\,\times \\ &\quad\times\left[2y(1-x)[2\vartheta(x-y)+1] - \frac{\omega^2}{2\varepsilon\kappa}\right][\boldsymbol{\nabla}_\rho V(R_{z-x})]\cdot[\boldsymbol{\nabla}_\rho V(R_{z-y})], \end{aligned} \tag{58}$$

where $R_s = r\sqrt{s^2+\rho^2}$. In this formula the terms, antisymmetric with respect to the substitution $\varepsilon \to \omega - \varepsilon$, $z \to 1 - z$ , are omitted, since their contribution to the integral vanishes. We emphasize, that in contrast to the first region, in the second one the integration over $z$ is performed in infinite limits. Changing the variables $z \to z + x$, $y \to y + x$ in triple integral over $x, y, z$ and performing the integration over $x$, we have

$$\begin{aligned} M_2 &= \frac{\alpha}{2\pi\omega}\int\limits_0^\omega d\varepsilon\int dr\, r^2 \exp\left[-i\frac{\omega r m^2}{2\varepsilon\kappa}\right]\int\limits_{\rho>\rho_0} d\boldsymbol{\rho}\int\limits_{-\infty}^{\infty} dz \int\limits_0^1 dy(1-y)\,\times \\ &\quad\times\left[\frac{4}{3}(1-y)^2 + 2y - \frac{\omega^2}{\varepsilon\kappa}\right][\boldsymbol{\nabla}_\rho V(R_z)]\cdot[\boldsymbol{\nabla}_\rho V(R_{z-y})]. \end{aligned} \tag{59}$$

At $r \ll r_c$ the potential $V(r) \approx -Z\alpha/r$. Therefore, it is convenient to present the contribution (59) as a sum $M_2 = M_2^c + \delta M$, where $M_2^c$ is the value of $M_2$ at $V(r) = V_c(r) = -Z\alpha/r$.

At $V = V_c$ the integrals over $r$ and $\varepsilon$ can be easily taken, and we find

$$M_2^c = -\frac{2i\alpha(Z\alpha)^2\omega}{m^2}\int\limits_{\rho_0}^{\infty} d\rho\rho^3\int\limits_{-\infty}^{\infty} dz\int\limits_0^1 dy(1-y)\frac{2(1-y)^2/9 + y/3 - 1}{[z^2+\rho^2]^{3/2}[(z-y)^2+\rho^2]^{3/2}}. \tag{60}$$

Using the Feynman parametrization of denominators

$$\frac{1}{(AB)^{3/2}} = \frac{8}{\pi}\int\limits_0^1 dv\frac{\sqrt{v(1-v)}}{[Av + B(1-v)]^3},$$

we take the integrals over $\rho$ and $z$, and then over $y$ and $v$ bearing in mind, that $\rho_0 \ll 1$. Finally, for this (Coulomb) contribution we obtain

$$M_2^c = -i\frac{28\alpha(Z\alpha)^2\omega}{9m^2}\left(\ln\frac{\rho_0}{2}+\frac{31}{21}\right) \tag{61}$$

The sum of the contributions (57) and (61) gives the known result for a pure Coulomb potential [17]:

$$M_c = i\frac{28\alpha(Z\alpha)^2\omega}{9m^2}\left[\ln\frac{2\omega}{m} - i\frac{\pi}{2} - \mathrm{Re}\psi(1+iZ\alpha) - C - \frac{109}{42}\right]. \tag{62}$$

In the term $\delta M$, connected to screening, the lower limit of integration over $\rho$ can be replaced by zero owing to the convergence of the integral at $\rho \to 0$. Using the momentum representation for potentials

$$V(r) = \int \frac{d\mathbf{p}}{(2\pi)^3} \mathrm{e}^{i\mathbf{pr}}\tilde{V}(p)\,,$$

we take the integrals over $\boldsymbol{\rho}$ and $z$. The result of this integration is proportional to $\delta(\mathbf{p}_1 - \mathbf{p}_2)$. Integrating over $\mathbf{p}_2$ and over the angles of vector $\mathbf{p}_1$, we find

$$\begin{aligned} \delta M &= \frac{\alpha}{2\pi^3\omega}\int\limits_0^\omega d\varepsilon \int dr\, r \exp\left[-i\frac{\omega r m^2}{2\varepsilon\kappa}\right]\int\limits_0^1 dy(1-y)\left[\frac{4}{3}(1-y)^2 + 2y - \frac{\omega^2}{\varepsilon\kappa}\right]\times \\ &\times \int\limits_0^\infty dp\left[p^4\tilde{V}^2(p) - (4\pi Z\alpha)^2\right]\left(\frac{\sin\zeta}{\zeta^3} - \frac{\cos\zeta}{\zeta^2}\right), \end{aligned} \tag{63}$$

where $\zeta = rpy$. Passing from the variable $r$ to $\zeta$, integrating by parts over $p$ and taking the integral over $\zeta$, we obtain

$$\begin{aligned} \delta M &= \frac{\alpha\omega}{2\pi^3 m^2}\int\limits_0^1 dx x(1-x)\int\limits_0^1 dy(1/y-1)\left[\frac{4}{3}(1-y)^2 + 2y - \frac{1}{x(1-x)}\right]\times \\ &\times \int\limits_0^\infty dp\,(\partial_p p^4\tilde{V}^2(p))\left[\frac{1}{\eta} + \frac{1}{2}\left(1-\frac{1}{\eta^2}\right)\ln\left(\frac{1+\eta}{1-\eta-i0}\right)\right], \end{aligned} \tag{64}$$

where the substitution $\varepsilon \to \omega x$ is made, $\eta = 2\omega x(1-x)py/m^2$. Now we make the change of variables $y \to y/(2x(1-x))$, change the order of integration over $x$ and $y$, and take the integral over $x$. Finally, the contribution to the amplitude of forward Delbruck scattering due to screening reads

$$\begin{aligned} \delta M &= -\frac{\alpha\omega}{18\pi^3 m^2}\int\limits_0^\infty dp\,(\partial_p p^4\tilde{V}^2(p))\int\limits_0^{1/2}\frac{dy}{y}\left[\frac{1}{\eta} + \frac{1}{2}\left(1-\frac{1}{\eta^2}\right)\ln\left(\frac{1+\eta}{1-\eta-i0}\right)\right]\times \\ &\times\left[(6y^2+7y+7)\sqrt{1-2y} + 3y(2y^2-3y-3)\ln\left(\frac{1+\sqrt{1-2y}}{1-\sqrt{1-2y}}\right)\right], \end{aligned} \tag{65}$$

where $\eta = \omega p y/m^2$. This formula holds for arbitrary form of a screened Coulomb potential. In some special cases it can be essentially simplified. For the potential $V(r) = -Z\alpha\exp(-\beta r)/r$, when $\tilde{V}(p) = -4\pi Z\alpha/(p^2+\beta^2)$, all integrals in (65) can be taken:

$$\delta M = \frac{4i\alpha(Z\alpha)^2\omega}{9m^2}\left[33 - 13\tau^2 + \frac{3}{2}\tau^4 + \frac{1}{2}\tau(24 - 13\tau^2 + 3\tau^4)L + \frac{3}{8}(8 - 9\tau^2 + \tau^6)L^2\right]\,;$$
$$\tau = \sqrt{1 + \frac{2im^2}{\omega\beta}}\quad,\quad L = \ln\left(\frac{\tau-1}{\tau+1}\right)\,. \tag{66}$$

For more realistic Moliere potential [18], we have

$$\tilde{V}(p) = -4\pi Z\alpha\sum_{n=1}^{3}\frac{\alpha_n}{p^2+\beta_n^2}\,, \tag{67}$$
$$\alpha_1 = 0.1,\quad \alpha_2 = 0.55,\quad \alpha_3 = 0.35,\quad \beta_n = \beta_0 b_n,$$
$$b_1 = 6,\quad b_2 = 1.2,\quad b_3 = 0.3,\quad \beta_0 = mZ^{1/3}/121.$$

In this case

$$\delta M = -\frac{4i\alpha(Z\alpha)^2\omega}{9m^2}\int\limits_0^{1/2}\frac{dy}{y}\left[(6y^2+7y+7)\sqrt{1-2y} + 3y(2y^2-3y-3)\times\right. \tag{68}$$
$$\left.\times\ln\left(\frac{1+\sqrt{1-2y}}{1-\sqrt{1-2y}}\right)\right]\left\{\sum_{n\neq k}\alpha_n\alpha_k\left[1 - \frac{2i}{\gamma_n+\gamma_k} - \frac{2}{\gamma_n^2-\gamma_k^2}\ln\left(\frac{\gamma_n+i}{\gamma_k+i}\right)\right] + \sum_n\alpha_n^2\frac{\gamma_n}{\gamma_n+i}\right\}\,,$$

where $\gamma_n = \omega\beta_n y/m^2$.

As known, the imaginary part of the amplitude of forward Delbruck scattering is connected with the total cross section $\sigma$ of the electron-positron pair production by photon in an external field via the relation $\sigma = \mathrm{Im}M/\omega$. For the Moliere potential, using (62) and (68), we obtain the following expression for $\sigma$:

$$\sigma = \frac{28\alpha(Z\alpha)^2}{9m^2}\left\{\ln\frac{2\omega}{m} - \mathrm{Re}\psi(1+iZ\alpha) - C - \frac{109}{42} - \right.$$
$$-\int\limits_0^{1/2}\frac{dy}{y}\left[(\frac{6}{7}y^2+y+1)\sqrt{1-2y} + \frac{3}{7}y(2y^2-3y-3)\ln\left(\frac{1+\sqrt{1-2y}}{1-\sqrt{1-2y}}\right)\right]\times$$
$$\left.\times\left[\sum_{n\neq k}\alpha_n\alpha_k\left[1 - \frac{1}{\gamma_n^2-\gamma_k^2}\ln\left(\frac{1+\gamma_n^2}{1+\gamma_k^2}\right)\right] + \sum_n\alpha_n^2\frac{\gamma_n^2}{1+\gamma_n^2}\right]\right\}, \tag{69}$$

which agrees with the results, known in literature (see, e.g. [19]) .

Let us discuss now the dependence of the real part of Delbruck amplitude from the photon energy $\omega$. When $\omega/m^2 \ll r_c$ the screening can be neglected and $\mathrm{Re}M = \mathrm{Re}M_c = 14\pi\alpha(Z\alpha)^2\omega/9m^2$. The linear growth of the real part with increasing $\omega$ gradually becomes weaker and for $\omega/m^2 \gg r_c$ we obtain from (62) and (65)

$$\mathrm{Re}M \approx \frac{\alpha}{2\pi^3}\ln^2\left(\frac{\omega}{m^2 r_c}\right)\int\limits_0^{\infty}\frac{dp}{p}\left(\partial_p p^4\tilde{V}^2(p)\right)\,. \tag{70}$$

Here only the term containing a higher degree of the large logarithm is kept. For the illustration the real part of the forward Delbruck amplitude is shown in Fig. 1 for $Z = 82$ and $V(r) = -Z\alpha \exp(-m\alpha Z^{1/3} r)/r$ as a function of $\omega$.

One of the basic mechanisms of elastic scattering of a photon is the Compton scattering on atomic electrons. For the forward scattering the amplitude of this process is real and does not depend on $\omega$: $M_{Comp} = -4\pi Z\alpha/m$. It is seen in Fig. 1, that the interference between the amplitudes of Compton and Delbruck forward scattering should be taken into account already at relatively small energies.

From the calculation of the Delbruck scattering amplitude we learn ones more, that the use of the eikonal approximation for the description of high-energy small-angle scattering processes without proper ground can lead to incorrect results. For instance, the Coulomb corrections in the imaginary part of the forward Delbruck scattering amplitude (and, therefore, in the total cross section of pair production) would be completely lost if we used the eikonal Green function in the calculation.

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Fig. 1. The real part of the forward Delbruck scattering amplitude for the potential $V(r) = -Z\alpha \exp(-m\alpha Z^{1/3} r)/r$ in units of $4\pi Z\alpha/m$, $Z = 82$ as a function of $\omega$.

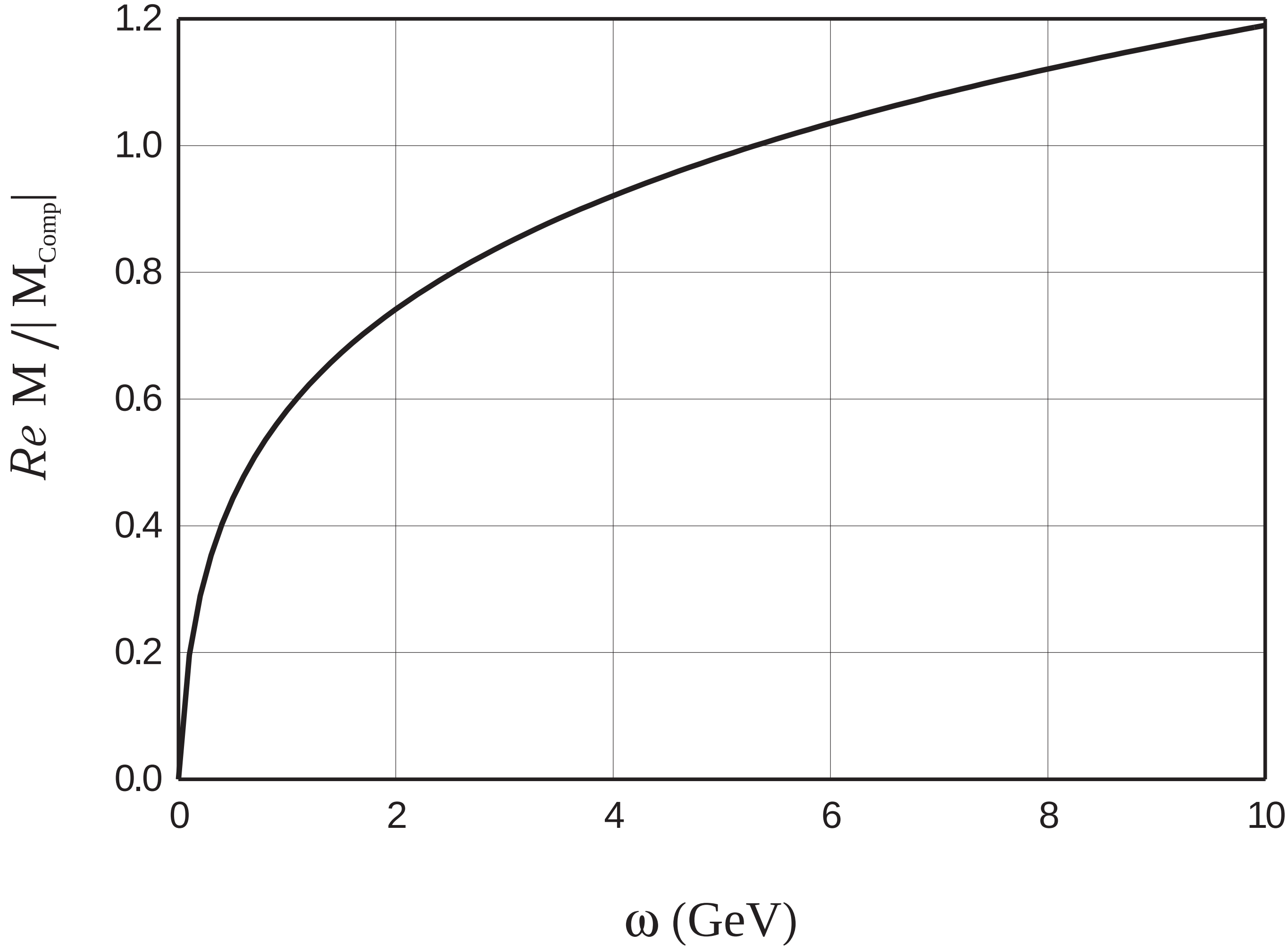


Fig. 1.